Green functions and nonlinear systems: Short time expansion

Marco Frasca*
Via Erasmo Gattamelata, 3
00176 Roma (Italy)
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We show that Green function methods can be straightforwardly applied to nonlinear equations appearing as the leading order of a short time expansion. Higher order corrections can be then computed giving a satisfactory agreement with numerical results. The relevance of these results relies on the possibility of fully exploiting a gradient expansion in both classical and quantum field theory granting the existence of a strong coupling expansion. Having a Green function in this regime in quantum field theory amounts to obtain the corresponding spectrum of the theory.

I. INTRODUCTION

Nonlinear equations represent a class of very difficult mathematical problems to manage by analytical methods. A lot of fundamental aspects of physics are described by these equations making not easy their understanding due to the lack of useful techniques.

In this paper we present an approach that is based on an unexpected result for Green functions. First hints in this direction were obtained in [1, 2] where we showed that for a part of the integration interval, a nonlinear differential equation like $\ddot{\phi} + \phi^3 = j$, being j a source term, can be solved by its Green function $\ddot{G} + G^3 = \delta(t)$ as $\phi(t) \approx \int_0^t G(t-t')j(t')dt'$.

Although this solution was put forward, the knowledge of this result is hardly useful unless we are not able to understand how to get higher order corrections. The aim of this paper is to give a proper understanding of this solution and to give a technique to get higher order corrections in order to improve it. We will show that it represents a short time solution. Then, a form factor described by polynomial terms in time can correct properly the propagator to improve in some cases this approximation.

The reason to give such a solution relies on the possibility to treat strong coupled quantum field theories that at the leading order produce nonlinear equations driven by a source. Strongly coupled theory can be managed by a gradient expansion that is the dual perturbation series to a weak coupling expansion as we proved in [1, 3, 4]. By "dual" we mean that two series can be obtained by simply interchanging the terms of the expansion producing in a case a series with an expansion parameter being the inverse of the expansion parameter of the other series, the asymptotic series so obtained holding in the proper limit where this parameter gives a converging expansion (coupling going to infinity in a case while going to zero in the other). This approach is true for any differential equation set and we applied it also in general relativity [3] obtaining a sound proof of the Belinski-Khalatnikov-Lifshitz conjecture [5, 6, 7] as this is a result of a gradient expansion.

A strongly coupled system in quantum mechanics is known to be a classical system as was firstly shown by Simon [8]. We revised this approach in [9] where we have seen that the gradient expansion for the Schrödinger equation, also known as Wigner-Kirkwood expansion, gives rise to a Thomas-Fermi approximation to the leading order for a many-body system [10] and has the same eigenvalue expansion as for a WKB approximation. Wigner-Kirkwood expansion is indeed the gradient expansion of the Schrödinger equation.

Gradient expansions in quantum field theories were not widely used before while their proper understanding is not that easy. Our aim in this paper is to fully exploit this perturbation approach and its application in quantum field theory wherever possible. This method may pave the way to manage analytically some problems that now appear difficult to manage also in a wide variety of fields where nonlinear equations are at the foundations.

The paper is structured in the following way. In section II we show how to derive a gradient expansion out of a duality principle in perturbation theory with the proper understanding of the expansion parameter. In section III we present the main motivation for this paper, that is the continuum limit of a scalar quantum field theory giving rise to a model nonlinear equation we will use throughout the paper. In section IV we present the method firstly applied to a simple Riccati equation having a known analytical solution and then we generalize our method to the case of the leading order equation of a scalar field theory. In section V we give the numerical results showing how the

^{*}marcofrasca@mclink.it

approximation improves with higher order corrections varying also the forcing term into the equation. In section VI we compare our approach with functional iteration method, a well known method used to analyze non-linear differential equations. This will give a proper understanding of the speed of convergence of our method. Finally, in section VII conclusions are given.

II. DUAL EXPANSION FOR NONLINEAR PDES

In order to make the paper self-contained we present here some material already given in Ref.[4]. We specialize the presentation to a $\lambda \phi^4$ model that is our reference model.

The Hamiltonian of the model is given by

$$H = \int d^{D-1}x \left[\frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi)^2 + V(\phi) \right]$$
 (1)

being D the spacetime dimensionality and $V(\phi) = \frac{1}{2}\phi^2 + \frac{\lambda}{4}\phi^4$ and we take the case of a single component for the sake of simplicity. Hamilton equations are

$$\partial_t \phi = \pi$$

$$\partial_t \pi = \nabla^2 \phi - \phi - \lambda \phi^3.$$
(2)

We can see at glance that we can chose to do perturbation theory by two different choices. One can take either $\lambda \phi^3$ or $\nabla^2 \phi - \phi$ as a small term. What we want to understand is the link between the two series with respect to the parameter λ .

By choosing $\lambda \phi^3$ as a small term one gets the small perturbation series

$$\partial_t \phi_0 = \pi_0$$

$$\partial_t \phi_1 = \pi_1$$

$$\partial_t \phi_2 = \pi_2$$

$$\vdots$$

$$\partial_t \pi_0 = \nabla^2 \phi_0 - \phi_0$$

$$\partial_t \pi_1 = \nabla^2 \phi_1 - \phi_1 - \phi_0^3$$

$$\partial_t \pi_2 = \nabla^2 \phi_2 - \phi_2 - 3\phi_0^2 \phi_1$$

$$\vdots$$

$$\vdots$$

$$\vdots$$

where it easily seen that the free theory, $\Box \phi_0 + \phi_0 = 0$, is the leading order solution. Our aim is to derive a dual perturbation series to this one meaning by this that we want a series with a development parameter going as $\frac{1}{\lambda}$. In order to reach our aim, following the principle of duality in perturbation theory [11] we put

$$\tau = \sqrt{\lambda}t$$

$$\pi = \sqrt{\lambda} \left(\pi_0 + \frac{1}{\lambda} \pi_1 + \frac{1}{\lambda^2} \pi_2 + \dots \right)$$

$$\phi = \phi_0 + \frac{1}{\lambda} \phi_1 + \frac{1}{\lambda^2} \phi_2 + \dots$$
(4)

The following non trivial set of equations is obtained

$$\partial_{\tau}\phi_{0} = \pi_{0}
\partial_{\tau}\phi_{1} = \pi_{1}
\partial_{\tau}\phi_{2} = \pi_{2}
\vdots
\partial_{\tau}\pi_{0} = -\phi_{0}^{3}
\partial_{\tau}\pi_{1} = \nabla^{2}\phi_{0} - \phi_{0} - 3\phi_{0}^{2}\phi_{1}
\partial_{\tau}\pi_{2} = \nabla^{2}\phi_{1} - \phi_{1} - 3\phi_{0}\phi_{1}^{2} - 3\phi_{0}^{2}\phi_{2}
\vdots$$
(5)

whose solution proves the existence of a dual perturbation series for the classical $\lambda \phi^4$ theory. We easily realize that this set of equations would have been obtained if one takes as a small term $\nabla^2 \phi - \phi$ giving rise in this case to a gradient expansion, that is a series having derivatives in space as small terms. So, strong coupling expansion and weak coupling expansion are related by the duality principle in perturbation theory [11] producing in the former case a gradient expansion. This result can be easily generalized to any kind of PDE [3, 4].

The point to be noted is that to have an analytical result for a strong coupling expansion we have to solve a nonlinear differential equation that in this case is given by

$$\partial_{\tau}^{2}\phi_{0} + \phi_{0}^{3} = 0. \tag{6}$$

Things can be more involved when a source term is present as is generally the case in quantum field theory and a meaning should be attached to the leading order equation

$$\partial_{\tau}^2 \phi_0 + \phi_0^3 = j. \tag{7}$$

being j a source term. The aim of this paper is to show that indeed an approach through Green functions is applicable in these cases, that is, as already shown by numerical methods in [1, 2], a first approximated solution is given by

$$\phi = \int_0^{\tau} d\tau' G(\tau - \tau') j(\tau') \tag{8}$$

being $G(\tau)$ the Green function solving the equation

$$\partial_{\tau}^{2} G(\tau) + G(\tau)^{3} = \delta(\tau). \tag{9}$$

We will give in this paper a general approach to compute higher order corrections to this result. We will note that the method can be applicable when a solution is known to an equation like

$$\partial_{\tau}^{2}G(\tau) + F(G(\tau)) = a\delta(\tau) \tag{10}$$

being a a constant and $F(G(\tau))$ a generic term. Otherwise we are not able to get analytical results and we have to resort to numerical methods. Anyhow, the situation is favorable for the most common models.

III. GRADIENT EXPANSION AND QUANTUM FIELD THEORY

Quantum field theory of the model we are considering is given by the generating functional

$$Z[j] = \int [d\phi] e^{\left\{i \int d^D x \left[\frac{1}{2} (\partial_t \phi)^2 - \frac{\lambda}{4} \phi^4 + j\phi\right]\right\}} e^{\left\{-i \int d^D x \left[\frac{1}{2} (\nabla \phi)^2 + \frac{1}{2} \phi^2\right]\right\}}$$
(11)

that we have written separating the spatial part from the rest. We did this in order to derive the strong coupling expansion to this case as already done in sec.II for the classical model. By doing the expansion, considering the gradient term $\frac{1}{2}(\nabla \phi)^2 + \frac{1}{2}\phi^2$ as small, the leading order term to be computed is

$$Z_0[j] = \int [d\phi] e^{\left\{i \int d^D x \left[\frac{1}{2} (\partial_t \phi)^2 - \frac{\lambda}{4} \phi^4 + j\phi\right]\right\}}$$
 (12)

and in the end we are left with the equation to solve

$$\partial_t^2 \phi + \lambda \phi^3 = j \tag{13}$$

that is the leading order of our gradient expansion as already seen in sec.II.

The applicability of the Green function method implies that also in a strong coupling regime one can obtain information on the spectrum of the theory in this limit. We can exploit this point easily for a our case. Firstly, we use the mass μ_0 of the theory to make all adimensional putting $x \to \mu_0 x$, $\phi^2 \to \mu_0^{2-D} \phi^2$ and introducing the coupling constant $g = \frac{\lambda}{\mu_0^{4-D}}$. Then, let us consider the equation

$$\partial_t^2 G + qG^3 = \delta(t) \tag{14}$$

that has solution [1]

$$G(t) = \theta(t) \left(\frac{2}{q}\right)^{\frac{1}{4}} \operatorname{sn}\left[\left(\frac{g}{2}\right)^{\frac{1}{4}} t, i\right]$$
(15)

being $\theta(t)$ the Heaviside function and sn a Jacobi elliptical function. Being the equation second order we have that also the time reversed solution holds. It is known [12] that the following series holds for this Jacobi function

$$\operatorname{sn}(u,i) = \frac{2\pi}{K(i)} \sum_{n=0}^{\infty} \frac{(-1)^n e^{-(n+\frac{1}{2})\pi}}{1 + e^{-(2n+1)\pi}} \sin\left[(2n+1)\frac{\pi u}{2K(i)}\right]$$
(16)

being $K(i) = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1+\sin^2\theta}} \approx 1.3111028777$ a constant. Then the mass spectrum of the theory in the limit of a very

large g is given by $E_n = (2n+1) \frac{\pi}{2K(i)} \left(\frac{g}{2}\right)^{\frac{1}{4}} \mu_0$ that we can recognize as the one of a harmonic oscillator.

So, the main physical motivation to study our approach through Green functions for nonlinear equations is to have a deeper understanding in quantum field theories but the method is rather general and could find applications in a lot of other fields.

IV. GREEN FUNCTION METHOD FOR NONLINEAR DIFFERENTIAL EQUATIONS

In order to make our approach as clearer as possible, we consider the trivial problem of a Riccati equation

$$\dot{y} + y^2 = 1 \tag{17}$$

with the initial condition y(0) = 0. The solution is given by $y(t) = \tanh(t)$. A Green function is easy to compute for this equation being given by

$$G(t) = \theta(t) \frac{1}{1+t}. (18)$$

So, let us consider the following small time expansion as a solution of the above Riccati equation

$$y(t) \approx \int_0^t dt' \frac{1}{1+t-t'} + a \int_0^t dt' \frac{1}{1+t-t'} (t-t')$$

$$+ b \int_0^t dt' \frac{1}{1+t-t'} (t-t')^2 + c \int_0^t dt' \frac{1}{1+t-t'} (t-t')^3 + \dots$$
(19)

being a, b and c constants to be computed. In order to compute these constants we consider the equation we started with and compute all the derivatives till the order we are interested in. Then, we compare these derivatives with the one obtained through equation (19) fixing in this way the values of the constants to make them equal. So, from eq.(19) one gets

$$y(t) \approx \ln(t+1) + a[-\ln(t+1) + t] + b[\ln(t+1) + \frac{t^2}{2} - t] + c[-\ln(t+1) + \frac{t^3}{3} + t - \frac{t^2}{2}] + \dots$$
 (20)

and from this we can compute y(0), $\dot{y}(0)$, $\ddot{y}(0)$ and so on. From the Riccati equation we have y(0)=0, $\dot{y}(0)=1$, $\ddot{y}(0)=0$ and so on giving finally a=1, b=-1 and c=-1 for our case yielding $y(t)=t-\frac{t^3}{3}$ that are the first two terms of the Taylor series of the $\tanh(t)$, the exact solution of the equation. From this exercise we learn that the series (19) is a small time series solution of the original equation and that the convergence may be really slow. It is a rather interesting aspect of this approach that the Green function method has such a way to be applied to nonlinear equations. The case we considered here is a rather trivial one but things are made more interesting for the case of a $\lambda\phi^4$ when we go to a numerical comparison.

We want to apply the above approach to the case of equation (13). So, let us seek a solution in the form (properly normalized by μ_0)

$$\phi(t) \approx \int_{0}^{t} \left(\frac{2}{g}\right)^{\frac{1}{4}} \operatorname{sn}\left[\left(\frac{g}{2}\right)^{\frac{1}{4}} (t - t'), i\right] j(t') dt'$$

$$+ a \int_{0}^{t} \left(\frac{2}{g}\right)^{\frac{1}{4}} \operatorname{sn}\left[\left(\frac{g}{2}\right)^{\frac{1}{4}} (t - t'), i\right] (t - t')^{4} j(t') dt'$$

$$+ b \int_{0}^{t} \left(\frac{2}{g}\right)^{\frac{1}{4}} \operatorname{sn}\left[\left(\frac{g}{2}\right)^{\frac{1}{4}} (t - t'), i\right] (t - t')^{6} j(t') dt' + \dots$$
(21)

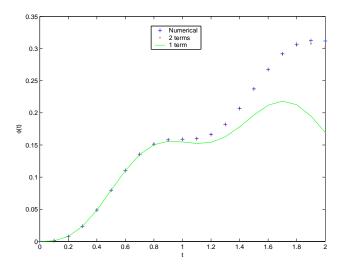


FIG. 1: Comparison for a driving source $j(t) = \sin(2\pi t)$.

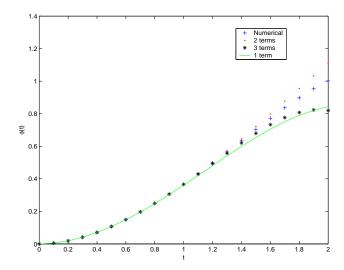


FIG. 2: Comparison for a driving source $j(t) = \exp(-t)$.

and after computing derivatives of this equation and eq.(13) we get easily $a = \frac{g}{20}$ and $b = -\frac{g}{56}[j(0)]^2$. This gives the result we aimed for. We have got the proper expansion by Green function method of a solution to a nonlinear differential equation. What we want to see is how good is this approximation when compared to numerical results. This will be shown in the following section.

V. NUMERICAL RESULTS

In order to verify the quality of our approximation we solve the equation (13) for two different driving sources and take the coupling constant g = 1. Firstly we considered $j(t) = \sin(2\pi t)$ and the results are given in fig.1. In this case we can only have a first order correction as j(0) = 0. The agreement is very satisfactory till the end of the integration interval.

The second case we considered $j(t) = e^{-t}$ permits to introduce another correction term but we notice no significant improvement due to the slow convergence of the approximation as can be seen from fig.2.

The quality of the approximation depends on j(t) that can make very demanding the need for higher order corrections.

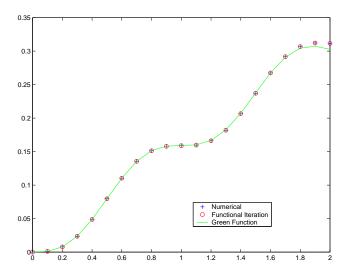


FIG. 3: Comparison between our approach and functional iteration method.

VI. COMPARISON WITH OTHER METHODS

There exist different techniques to manage non-linear differential equations. In order to have a proper comparison we have to limit our analysis to small time methods. In this sense, the most similar approach to ours is the functional iteration method [13]. This method proves to have a rapid convergence to a good approximant of the true solution when the equation is not stiff. This is exactly our case. So, let us consider the equation

$$\ddot{\phi}(t) + \phi^3(t) = \sin(2\pi t) \tag{22}$$

where dot means a time derivative. Functional iteration method implies that we solve the above equation iteratively. We assume $\phi(0) = 0$ and $\dot{\phi}(0) = 0$. We take as zero order iterate $\phi_0(t) = \phi(0) = 0$ and then for the successive iterates we have

$$\ddot{\phi}_{\nu+1}(t) = -\phi_{\nu}^3(t) + \sin(2\pi t) \tag{23}$$

starting with $\nu = 0$. Already at the second iterate we get a very good approximation to the true solution in the range we are considering. Then, we can compare this approximation with our method considering two terms. The results are presented in fig.3.

This result shows that functional iteration method has a faster convergence and at least another term should be computed with our approach to reach an identical precision in the required range. This result should also be expected on the ground of efficiency of iterative methods with respect to series solutions. So, in order to decide the proper method to use one should properly analyze the problem at hand.

VII. CONCLUSIONS

We have shown an approximation method to solve nonlinear differential equations using Green function methods. This method proves to be a small time expansion and the convergence in some cases may turn out really slow. The main point to be emphasized is the unexpected utility of this approach generally assumed to hold only for linear differential equations. This implies that a gradient expansion for nonlinear PDE can also be applied successfully and a quantum field theory obtained. In this latter case one should consider that a gradient expansion is a strong coupling expansion and then, information in this regime of the corresponding quantum field theory is given. This yields another method to approach these problems generally very difficulty to manage with analytical methods.

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